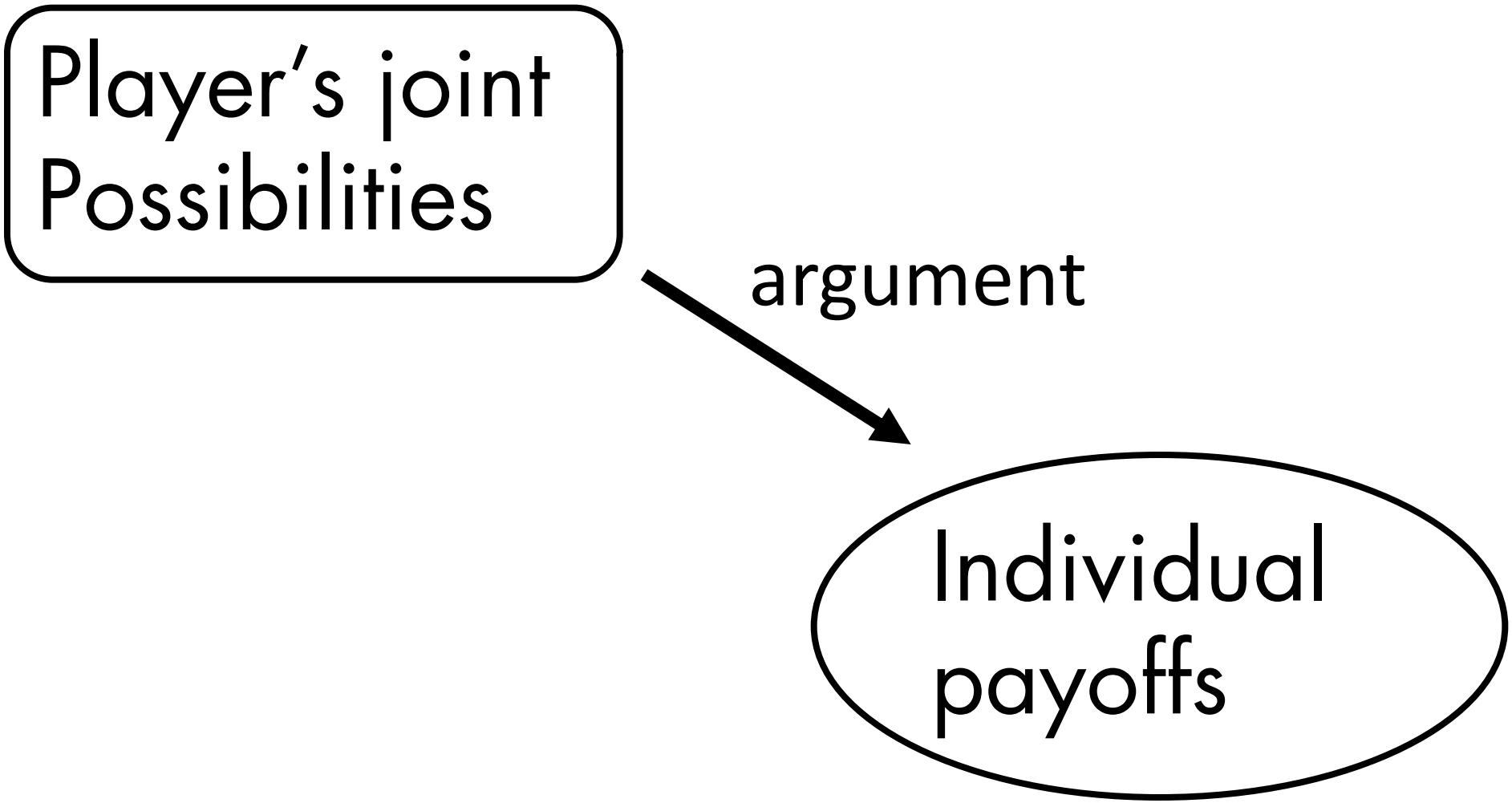
Decomposing R^2 with the Owen Value

Frank Hüttner & Marco Sunder
(Uni Leipzig)

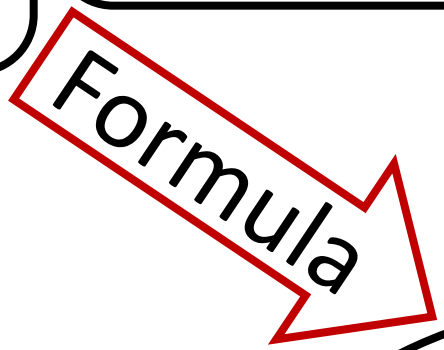
Game Theory


Player's joint Possibilities
argument
Individual payoffs

Cooperative Game Theory

Player's joint Possibilities

Assume: they work together



Individual payoffs

Cooperative Game Theory

Player's joint Possibilities

Assume: they work together

Formula

Shapley value

Individual payoffs

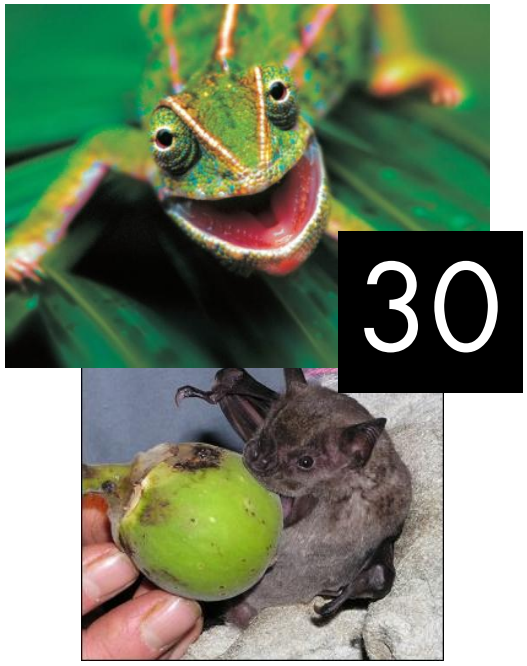
The Zoo

- Ape Bat Chameleon

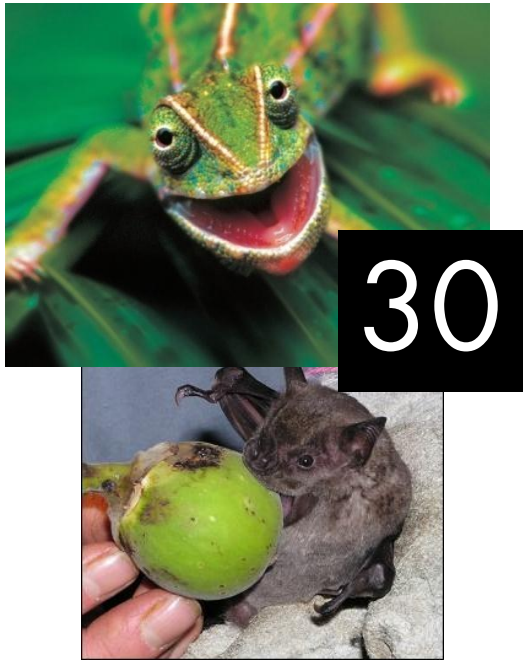


- 72 visitors/day = income
- How to share?

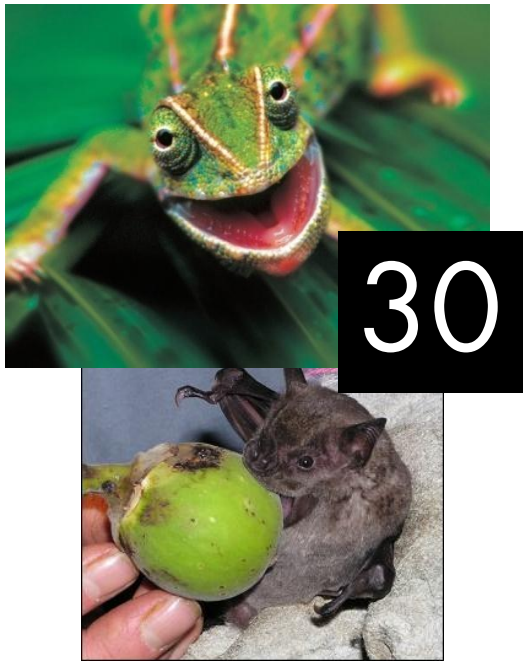
All together: 72



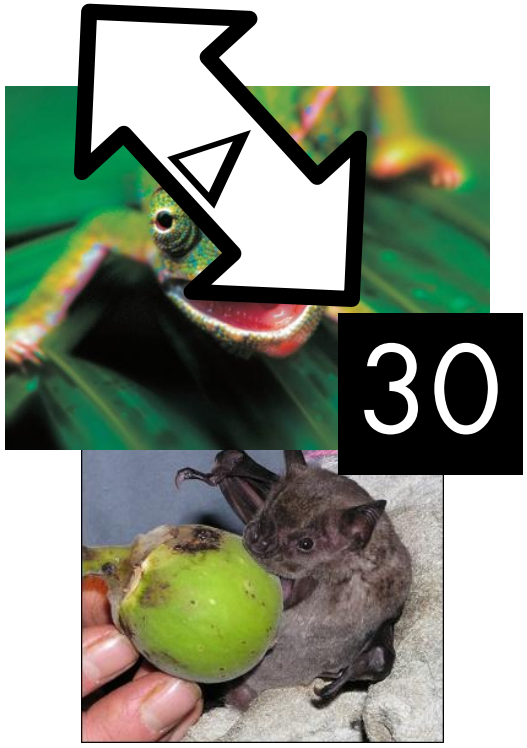
$Sh_A = 34$ **$Sh_B = 16$** **$Sh_C = 22$**

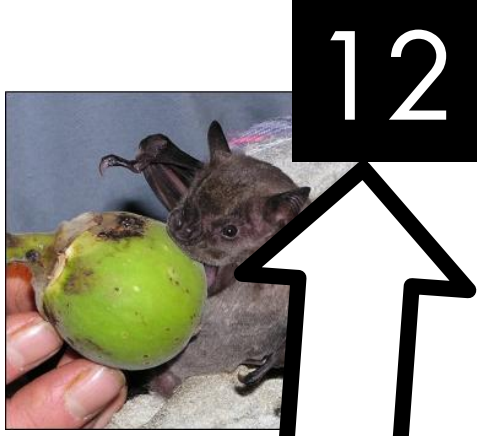


$Sh_A = 34$ $Sh_B = 16$ $Sh_C = 22$



All together: 72





All together: 72

24

12

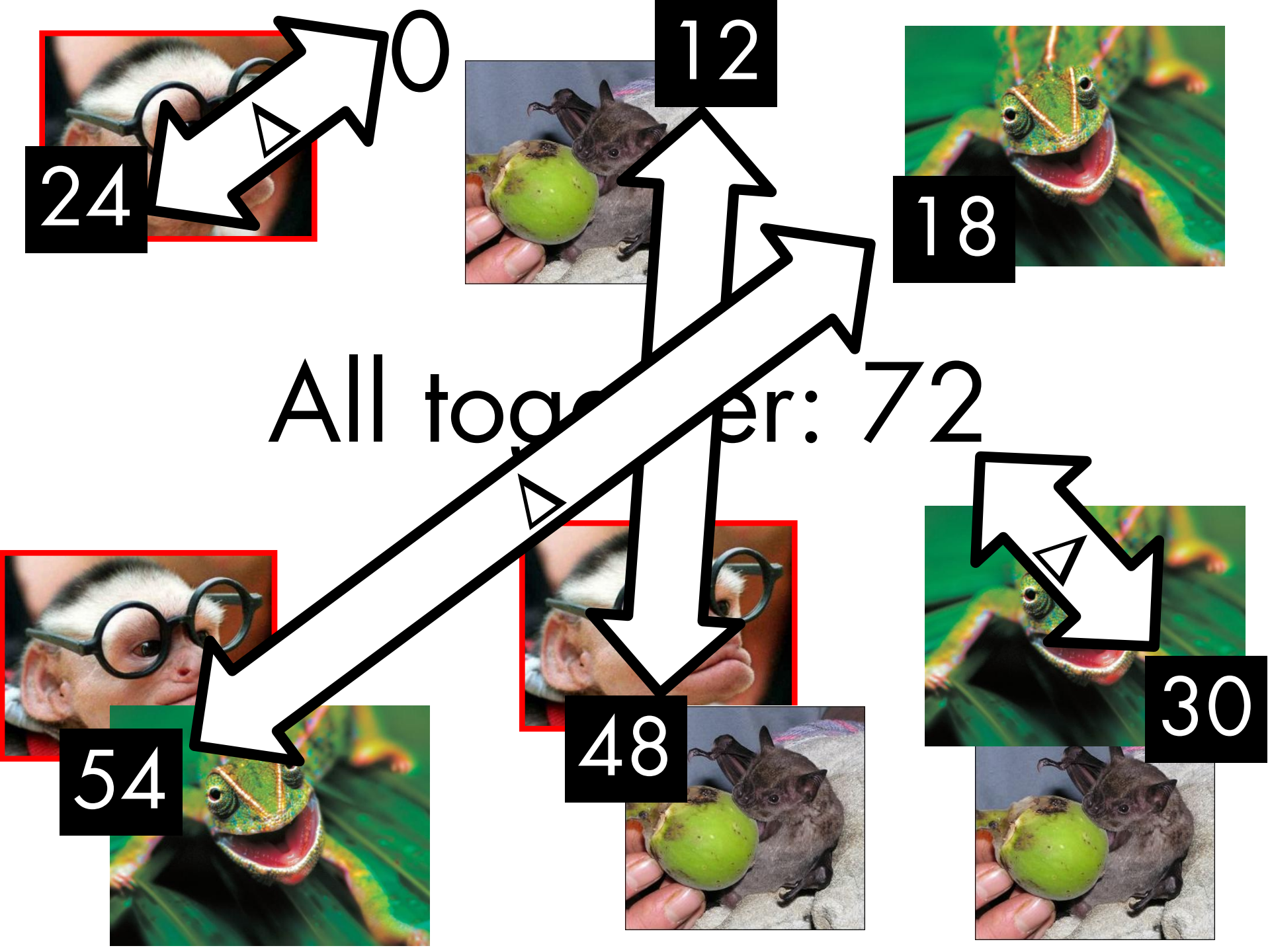
18

All together: 72

54

48

30


0
12
24
18
All together: 72
54
48
30

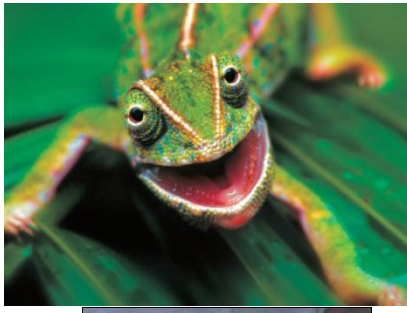
Lloyd S. Shapley

1. List all rank orderings of players

r
A, B, C
A, C, B
B, A, C
B, C, A
C, A, B
C, B, A







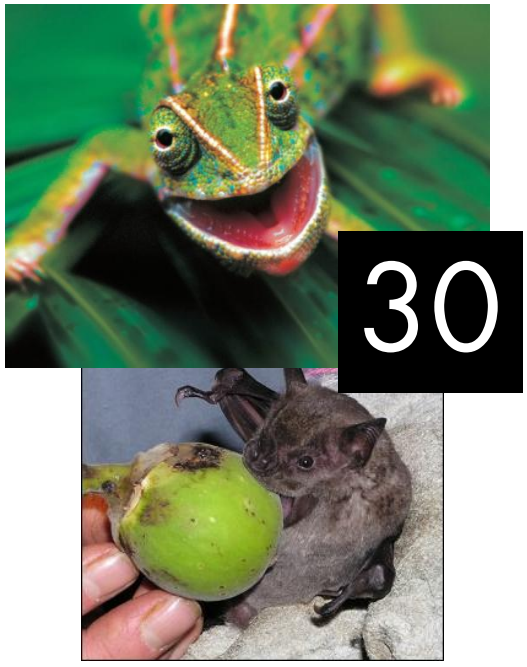

2. Take from ABC in this order

r		**-1st**	**..**	
A, B, C	ABC	BC	C	∅
A, C, B	ABC	BC	B	∅
B, A, C	ABC	AC	C	∅
B, C, A	ABC	AC	A	∅
C, A, B	ABC	AB	B	∅
C, B, A	ABC	AB	A	∅

3. Fill in worths

r		**-1st**	**..**	
A, B, C	ABC	BC	C	∅
A, C, B	ABC	BC	B	∅
B, A, C	ABC	AC	C	∅
B, C, A	ABC	AC	A	∅
C, A, B	ABC	AB	B	∅
C, B, A	ABC	AB	A	∅

All together: 72



3. Fill in worths

r		**-1st**	**..**	
A, B, C	Z2	BC	C	∅
A, C, B	ABC	BC	B	∅
B, A, C	ABC	AC	C	∅
B, C, A	ABC	AC	A	∅
C, A, B	ABC	AB	B	∅
C, B, A	ABC	AB	A	∅

3. Fill in worths

r		**-1st**	**..**	
A, B, C	72	BC	C	$\underline{\varnothing}$
A, C, B	ABC	BC	B	$\varnothing$
B, A, C	ABC	AC	C	$\varnothing$
B, C, A	ABC	AC	A	$\varnothing$
C, A, B	ABC	AB	B	$\varnothing$
C, B, A	ABC	AB	A	$\varnothing$

3. Fill in worths

r		**-1st**	**..**	
A, B, C	72	BC	C	0
A, C, B	ABC	BC	B	∅
B, A, C	ABC	AC	C	∅
B, C, A	ABC	AC	A	∅
C, A, B	ABC	AB	B	∅
C, B, A	ABC	AB	A	∅

3. Fill in worths

r		**-1st**	**..**	
A, B, C	72	BC	C	0
A, C, B	ABC	BC	B	∅
B, A, C	ABC	AC	C	∅
B, C, A	ABC	AC	A	∅
C, A, B	ABC	AB	B	∅
C, B, A	ABC	AB	A	∅

3. Fill in worths

r		**-1st**	**..**	
A, B, C	72	BC	C	0
A, C, B	72	BC	B	0
B, A, C	72	AC	C	0
B, C, A	72	AC	A	0
C, A, B	72	AB	B	0
C, B, A	72	AB	A	0

3. Fill in worths

r		**-1st**	**..**	
A, B, C	72	BC	C	0
A, C, B	72	BC	B	0
B, A, C	72	AC	C	0
B, C, A	72	AC	A	0
C, A, B	72	AB	B	0
C, B, A	72	AB	A	0

3. Fill in worths

r		**-1st**	**..**	
A, B, C	72	BC	C	0
A, C, B	72	BC	B	0
B, A, C	72	AC	C	0
B, C, A	72	AC	A	0
C, A, B	72	AB	B	0
C, B, A	72	AB	A	0

All together: 72

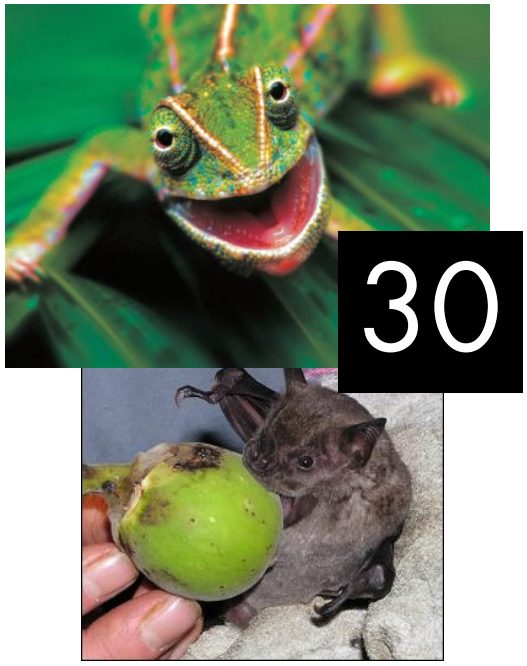
3. Fill in worths

r		**-1st**	**..**	
A, B, C	72	30	C	0
A, C, B	72	BC	B	0
B, A, C	72	AC	C	0
B, C, A	72	AC	A	0
C, A, B	72	AB	B	0
C, B, A	72	AB	A	0

3. Fill in worths

r		**-1st**	**..**	
A, B, C	72	30	C	0
A, C, B	72	BC	B	0
B, A, C	72	AC	C	0
B, C, A	72	AC	A	0
C, A, B	72	AB	B	0
C, B, A	72	AB	A	0

All together: 72



3. Fill in worths

r		**-1st**	**..**	
A, B, C	72	30	18	0
A, C, B	72	BC	B	0
B, A, C	72	AC	C	0
B, C, A	72	AC	A	0
C, A, B	72	AB	B	0
C, B, A	72	AB	A	0

3. Fill in worths

r		**-1st**	**..**	
A, B, C	72	30	18	0
A, C, B	72	..	..	0
B, A, C	72	..	..	0
B, C, A	72	..	..	0
C, A, B	72	..	..	0
C, B, A	72	..	..	0

3. Fill in worths

r		**-1st**	**..**	
A, B, C	72	30	18	0
A, C, B	72	..	..	0
B, A, C	72	..	..	0
B, C, A	72	..	..	0
C, A, B	72	..	..	0
C, B, A	72	..	..	0

24

12

18

All together: 72

54

48

30

3. Fill in worths

r		**-1st**	**..**	
A, B, C	72	30	18	0
A, C, B	72	..	..	0
B, A, C	72	..	..	0
B, C, A	72	54	24	0
C, A, B	72	..	..	0
C, B, A	72	..	..	0

4. Determine marginal contributions

r		-1st	..				
A, B, C	72	30	18	0	42	12	18
A, C, B	72	..	..	0	..	..	..
B, A, C	72	..	..	0	..	..	..
B, C, A	72	54	24	0	24	18	30
C, A, B	72	..	..	0	..	..	..
C, B, A	72	..	..	0	..	..	..

4. Determine marginal contributions



r		-1st	..				
A, B, C	72	30	18	0	42	12	18
A, C, B	72	30	..	0	42	..	..
B, A, C	72	54	18	0	36	..	..
B, C, A	72	54	24	0	24	18	30
C, A, B	72	48	12	0	36	..	..
C, B, A	72	48	24	0	24	..	..

r	
A, B, C	42
A, C, B	42
B, A, C	36
B, C, A	24
C, A, B	36
C, B, A	24
Σ	204

5. Take the average

r	
A, B, C	42
A, C, B	42
B, A, C	36
B, C, A	24
C, A, B	36
C, B, A	24
Σ	204

5. Take the average

$$Sh_A = \frac{204}{6}$$

$$= 34$$

r	
A, B, C	42
A, C, B	42
B, A, C	36
B, C, A	24
C, A, B	36
C, B, A	24
Σ	204

Shapley formula

- go over all rank orders $\mathbf{r} \in \boldsymbol{R}$

r	
A, B, C	42
A, C, B	42
B, A, C	36
B, C, A	24
C, A, B	36
C, B, A	24
Σ	204

Shapley formula

- go over all rank orders $\mathbf{r} \in \boldsymbol{R}$
- take average

Marginal Contributions $MC_A(\mathbf{r})$

r	
A, B, C	42
A, C, B	42
B, A, C	36
B, C, A	24
C, A, B	36
C, B, A	24
$\sum$	204

Shapley formula

- go over all rank orders $\mathbf{r} \in \boldsymbol{R}$
- take average

Marginal Contributions $MC_A(\mathbf{r})$

$$\frac{1}{\#\boldsymbol{R}} \cdot \sum_{\mathbf{r} \in \boldsymbol{R}} MC_A(\mathbf{r})$$

Why Shapley value?

$$Sh_A = \frac{1}{\#\boldsymbol{R}} \cdot \sum_{\mathbf{r} \in \boldsymbol{R}} MC_A(\mathbf{r})$$

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- If player improves on MCs

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- If player improves on MCs
  $\Rightarrow$ gets more

Why Shapley value?

$$Sh_A = \frac{1}{\#\boldsymbol{R}} \cdot \sum_{\mathbf{r} \in \boldsymbol{R}} MC_A(\mathbf{r})$$

- If player improves on MCs
  $\Rightarrow$ gets more

Monotonicity:
$MC_A \Uparrow \Rightarrow$ outcome $\Uparrow$

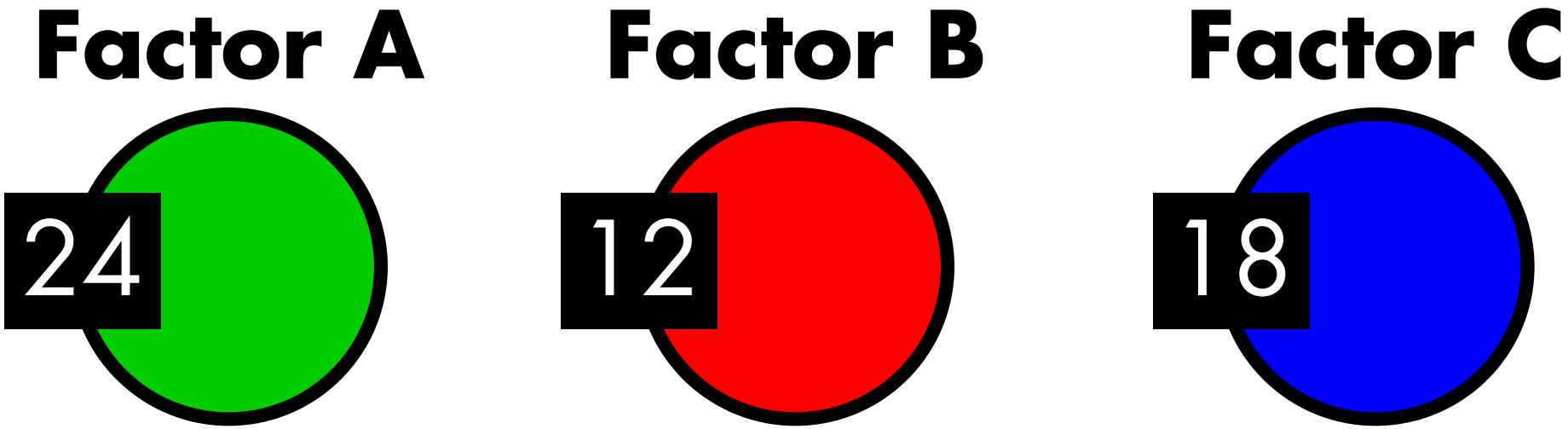
Why Shapley value?

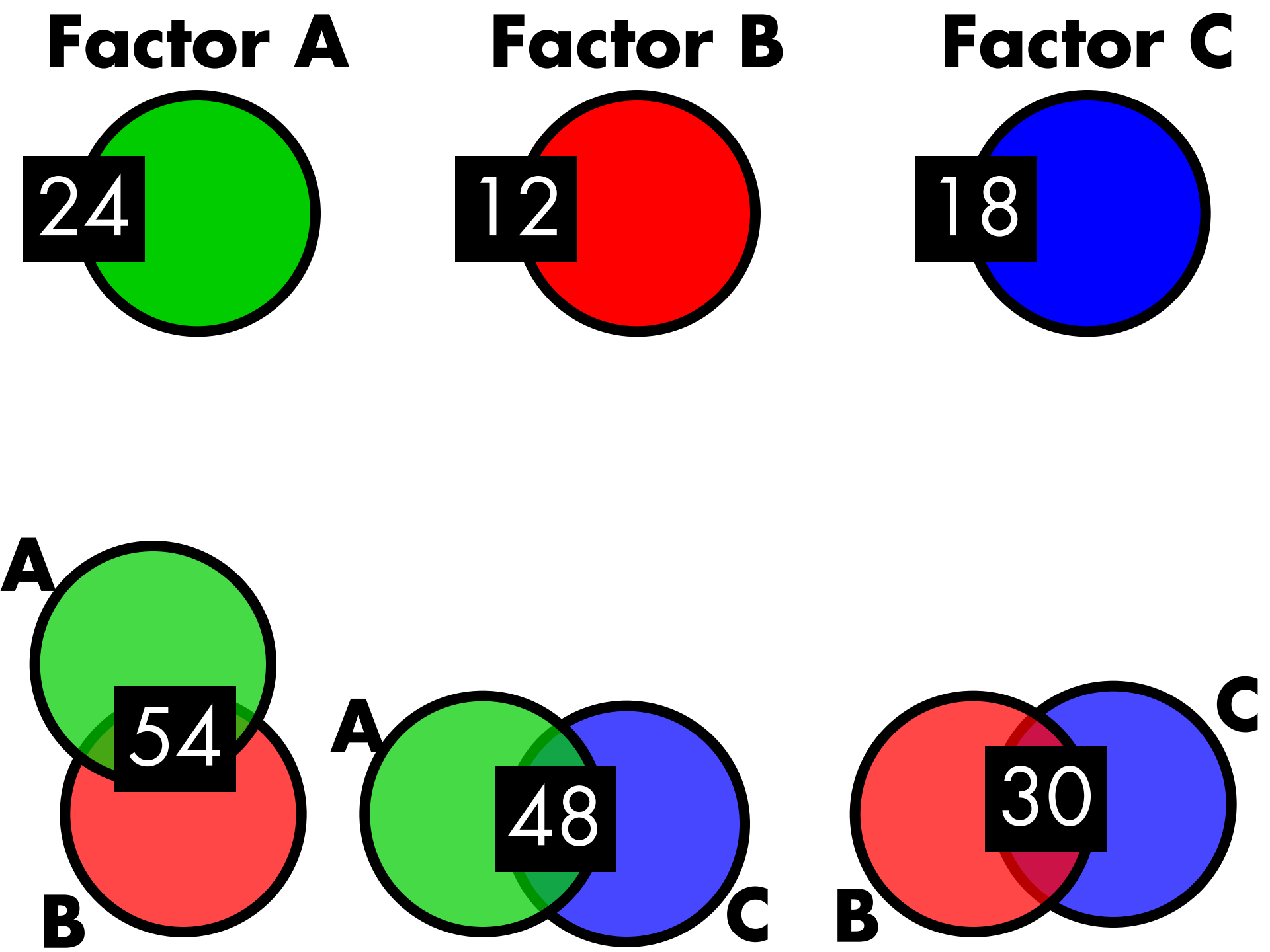
THEOREM (Young 1985):
Only one formula satisfies Monotonicity.
It is the Shapley value.

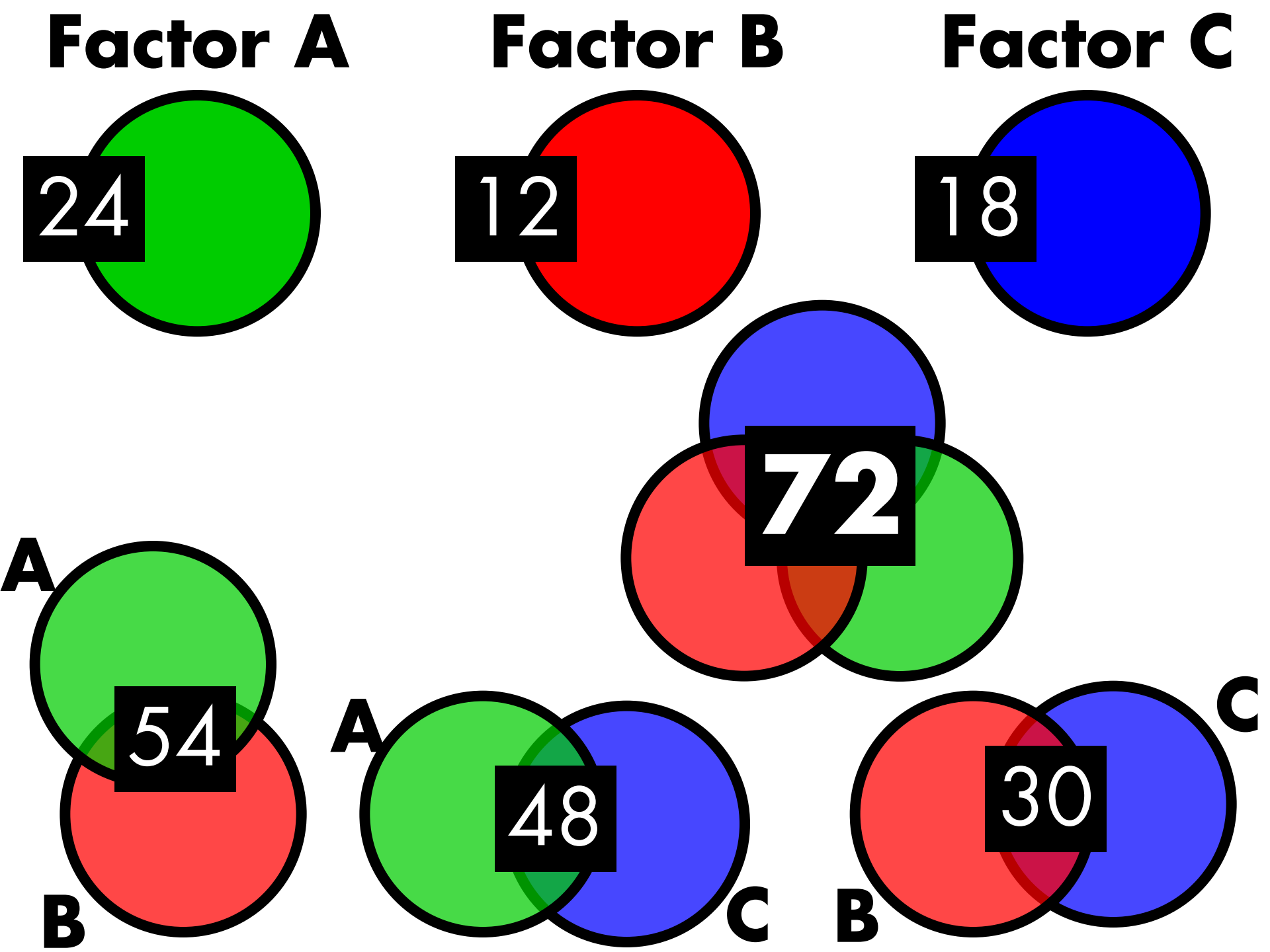
Why Shapley value?

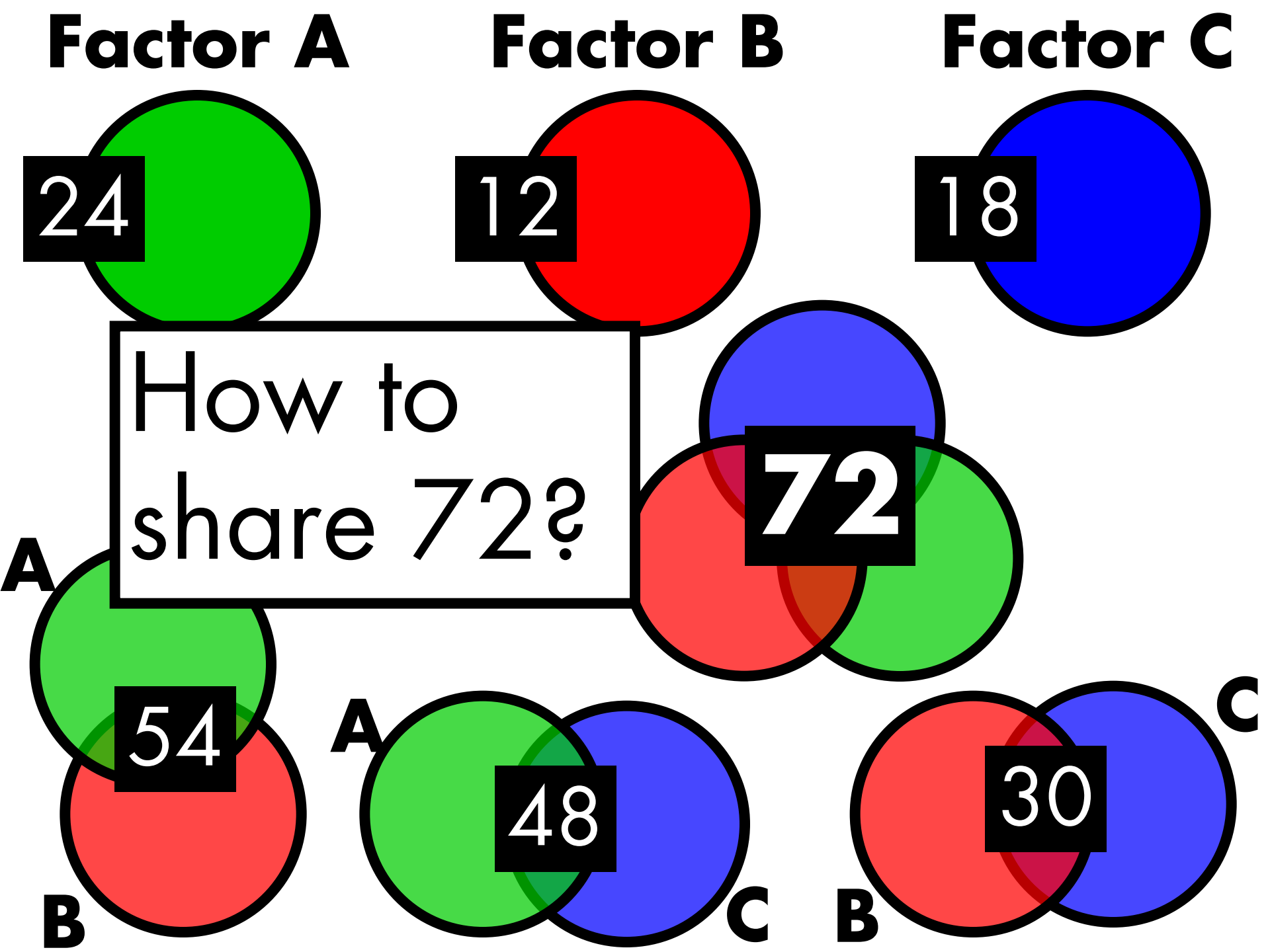
THEOREM (Young 1985):
Only one formula satisfies Monotonicity (*).
It is the Shapley value.

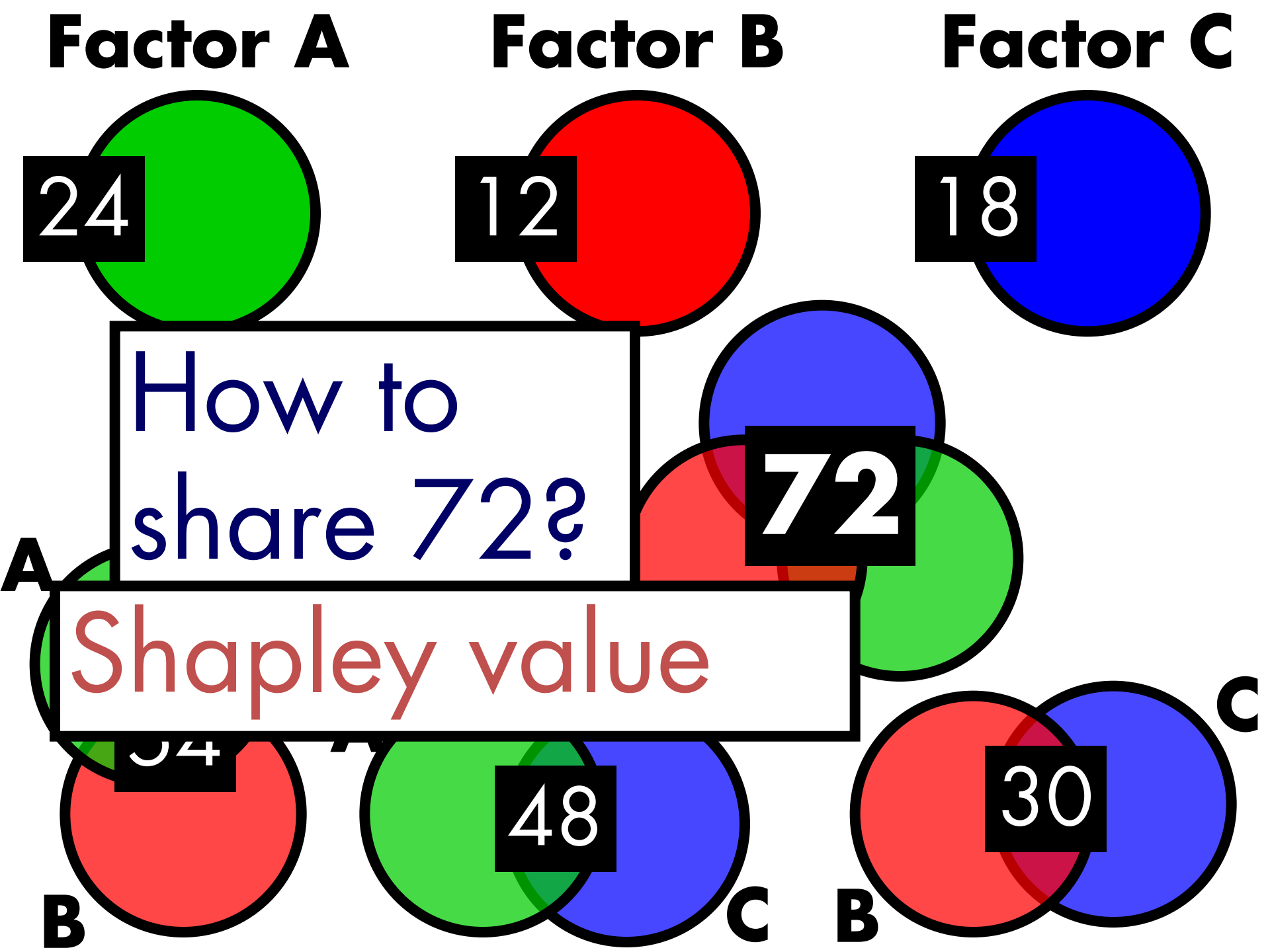
(*) and treats equal players equally

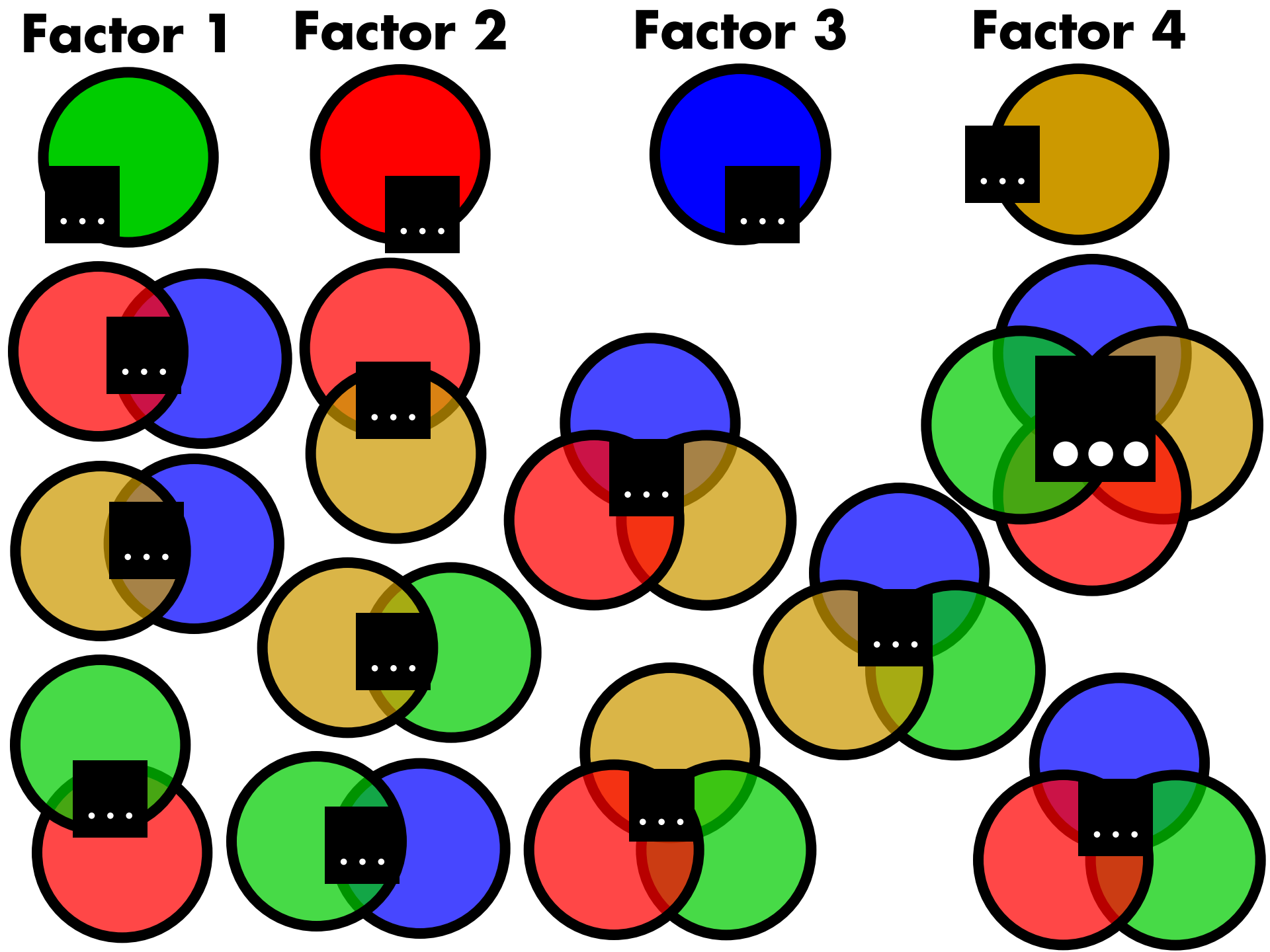

Factor A
24
Factor B
12
Factor C
18

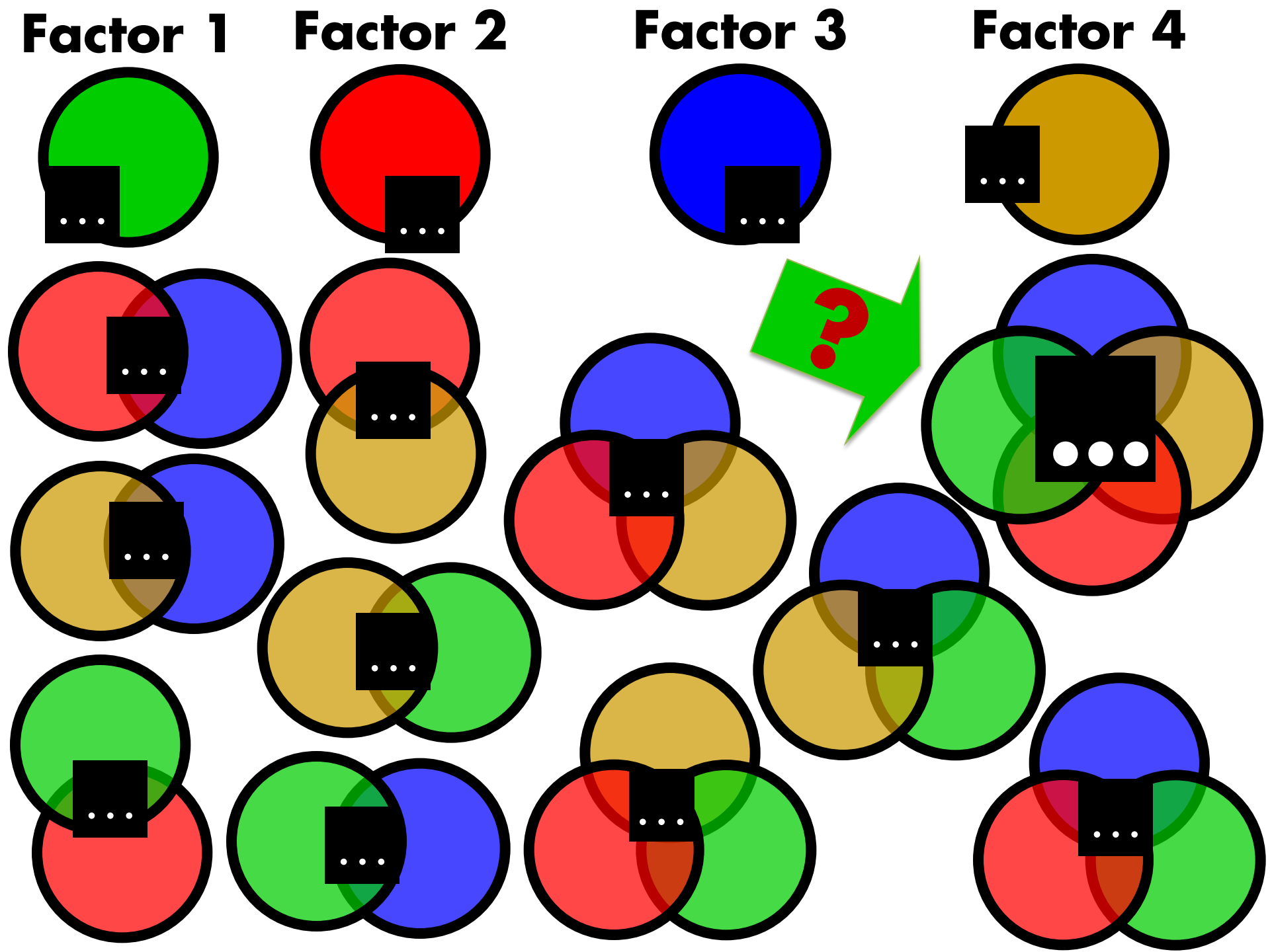

Factor A
Factor B
Factor C
24
12
18
A
54
B
A
48
C
30
C
B

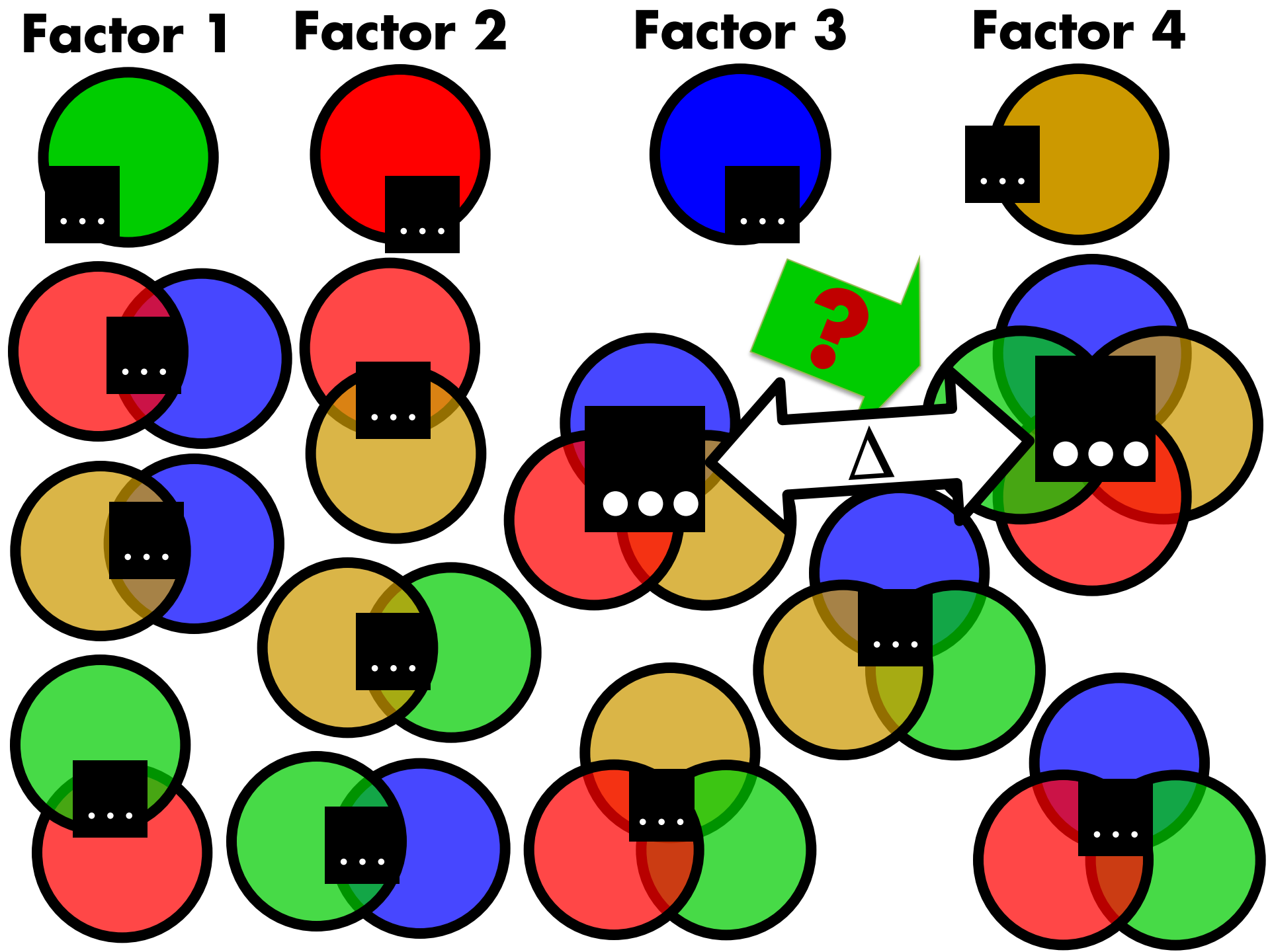

Factor A
Factor B
Factor C
24
12
18
72
A
54
B
A
48
C
30
C
B

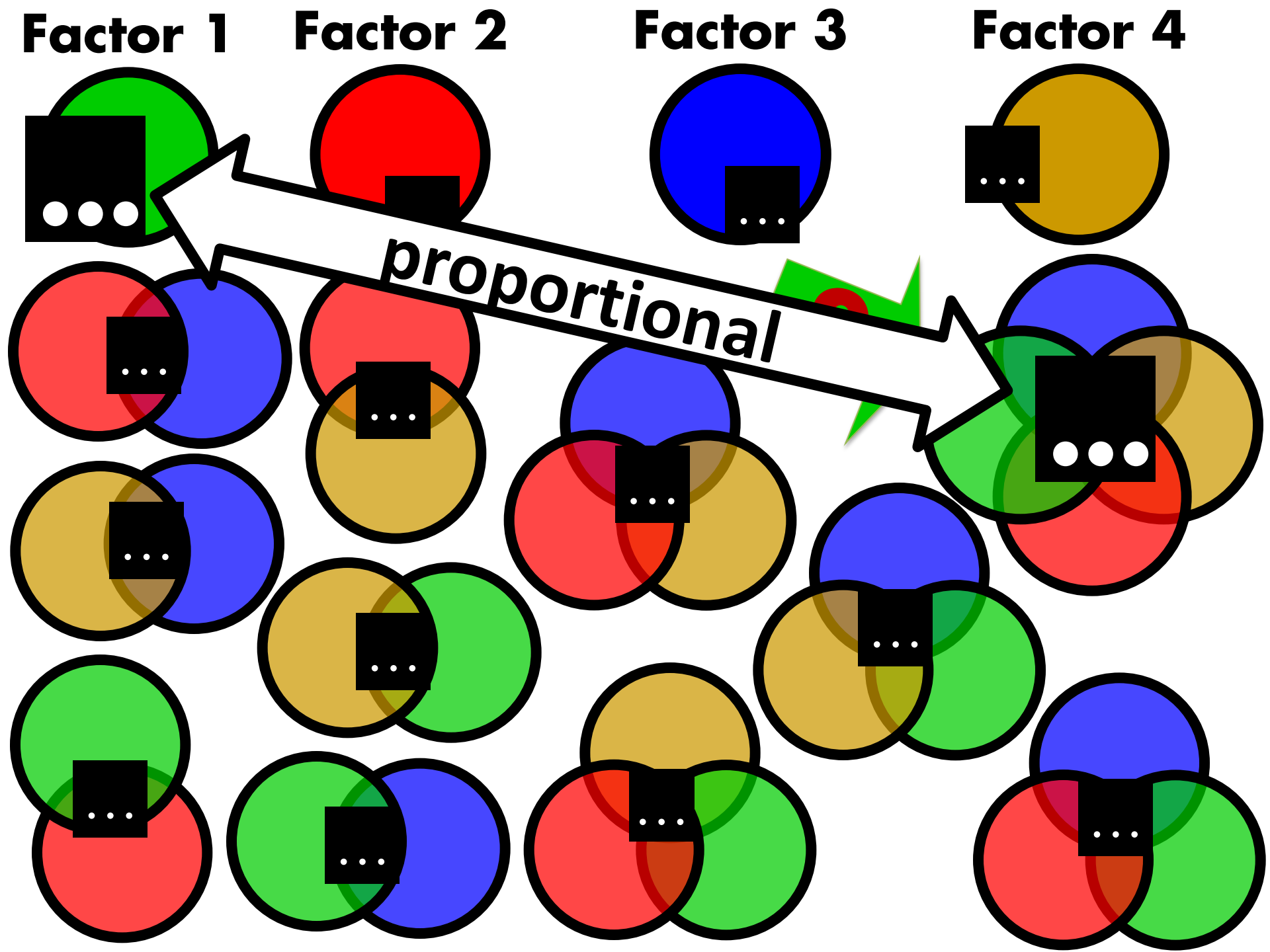

Factor A
Factor B
Factor C
24
12
18
How to share 72?
72
A
54
B
A
48
C
30
C
B

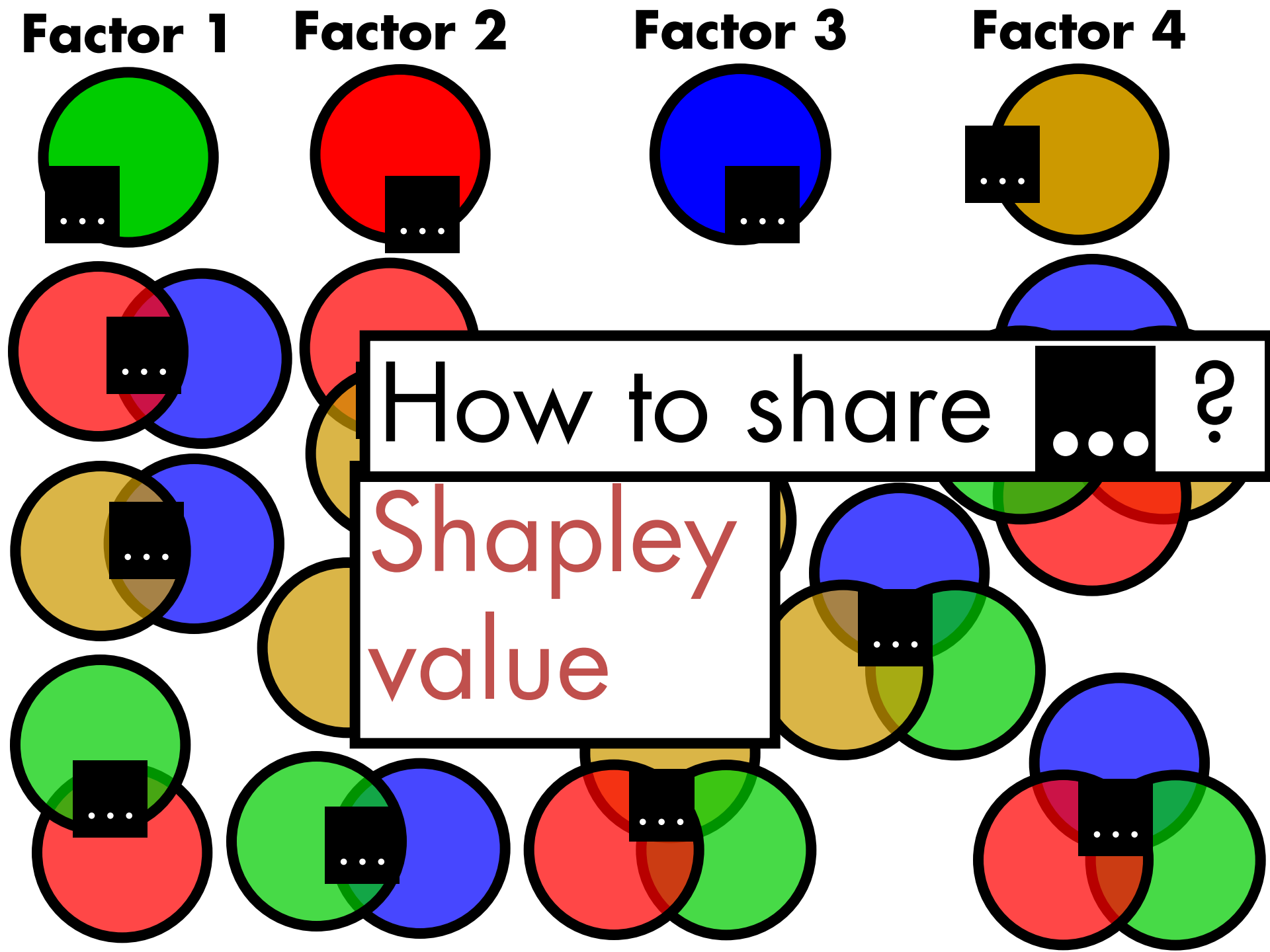

Factor A
Factor B
Factor C
24
12
18
How to share 72?
72
Shapley value
A
B
C
48
30


Factor 1
Factor 2
Factor 3
Factor 4


Factor 1
Factor 2
Factor 3
Factor 4
?


Factor 1
Factor 2
Factor 3
Factor 4
?


Factor 1
Factor 2
Factor 3
Factor 4
proportional


Factor 1
Factor 2
Factor 3
Factor 4
How to share ... ?
Shapley value

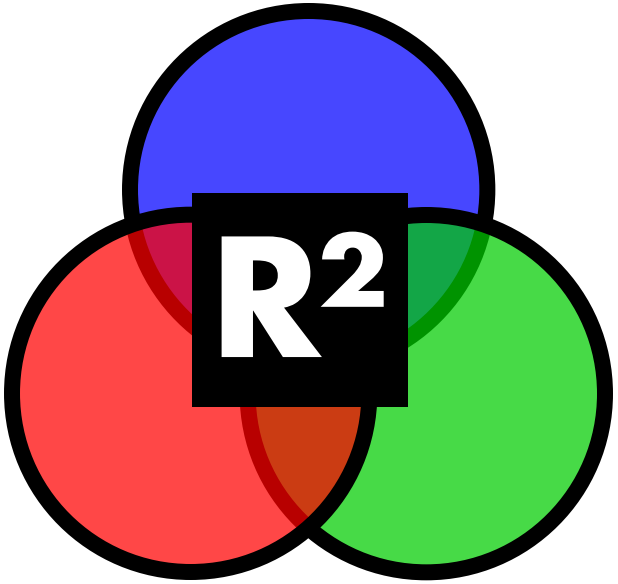
Linear regression

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon$$

$$\Rightarrow R^2$$

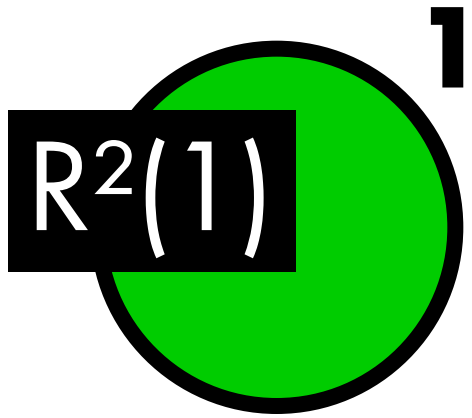
How to decompose R^2

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon$$

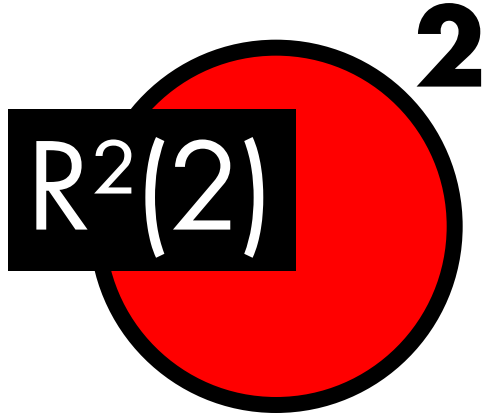


How to decompose R^2

$$y = \beta_0 + \beta_1 x_1 + \varepsilon$$

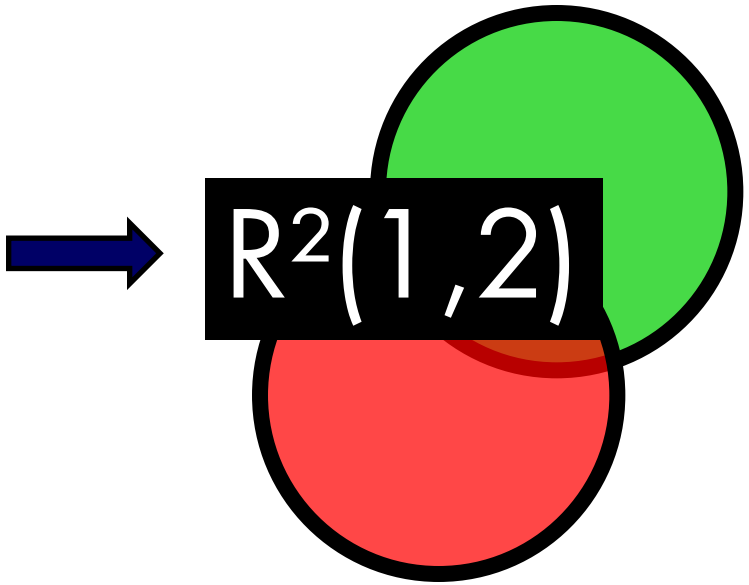


$$y = \beta_0 + \beta_2 x_2 + \varepsilon$$



How to decompose R^2

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$



How to decompose R^2

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon$$

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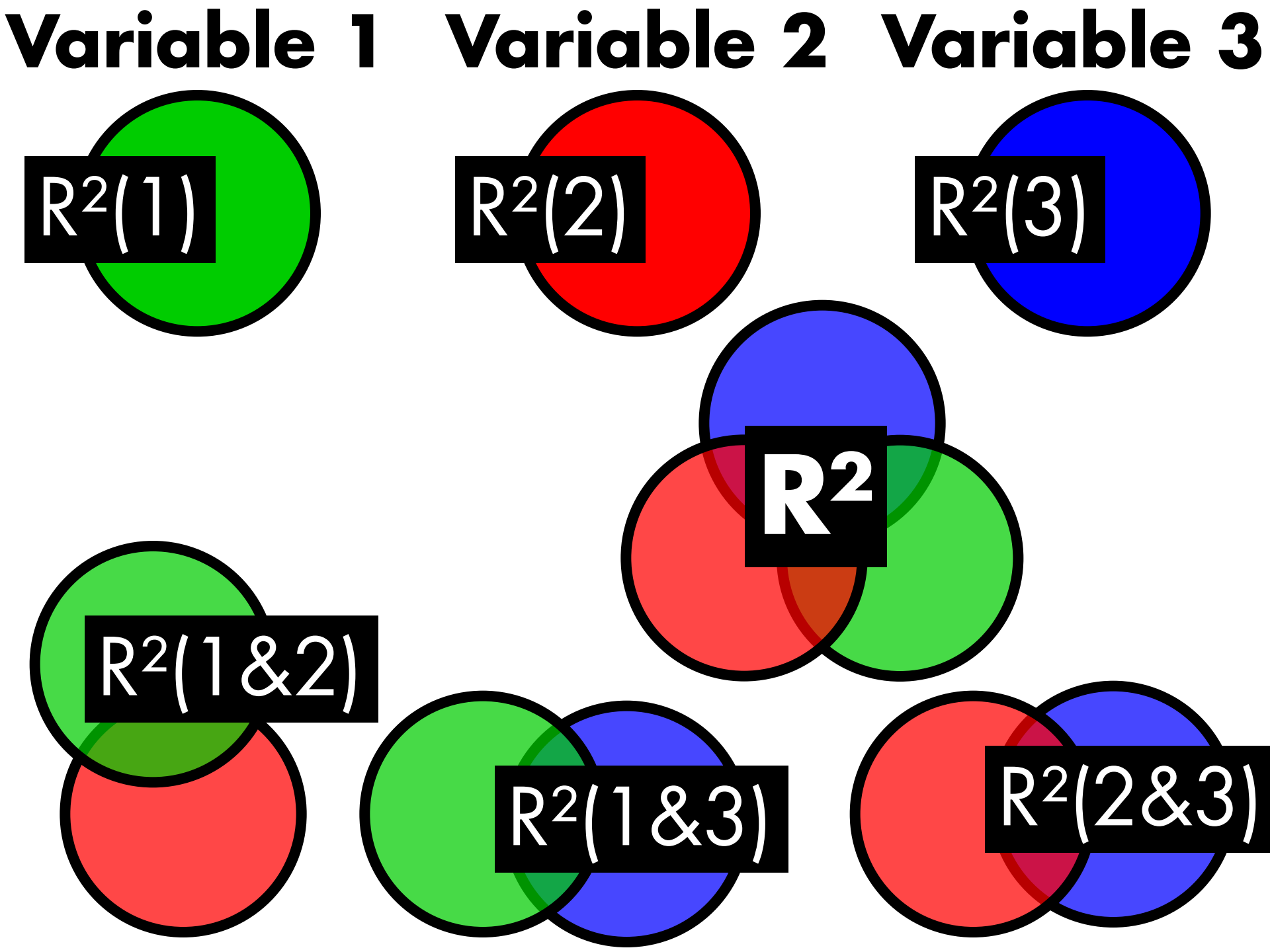
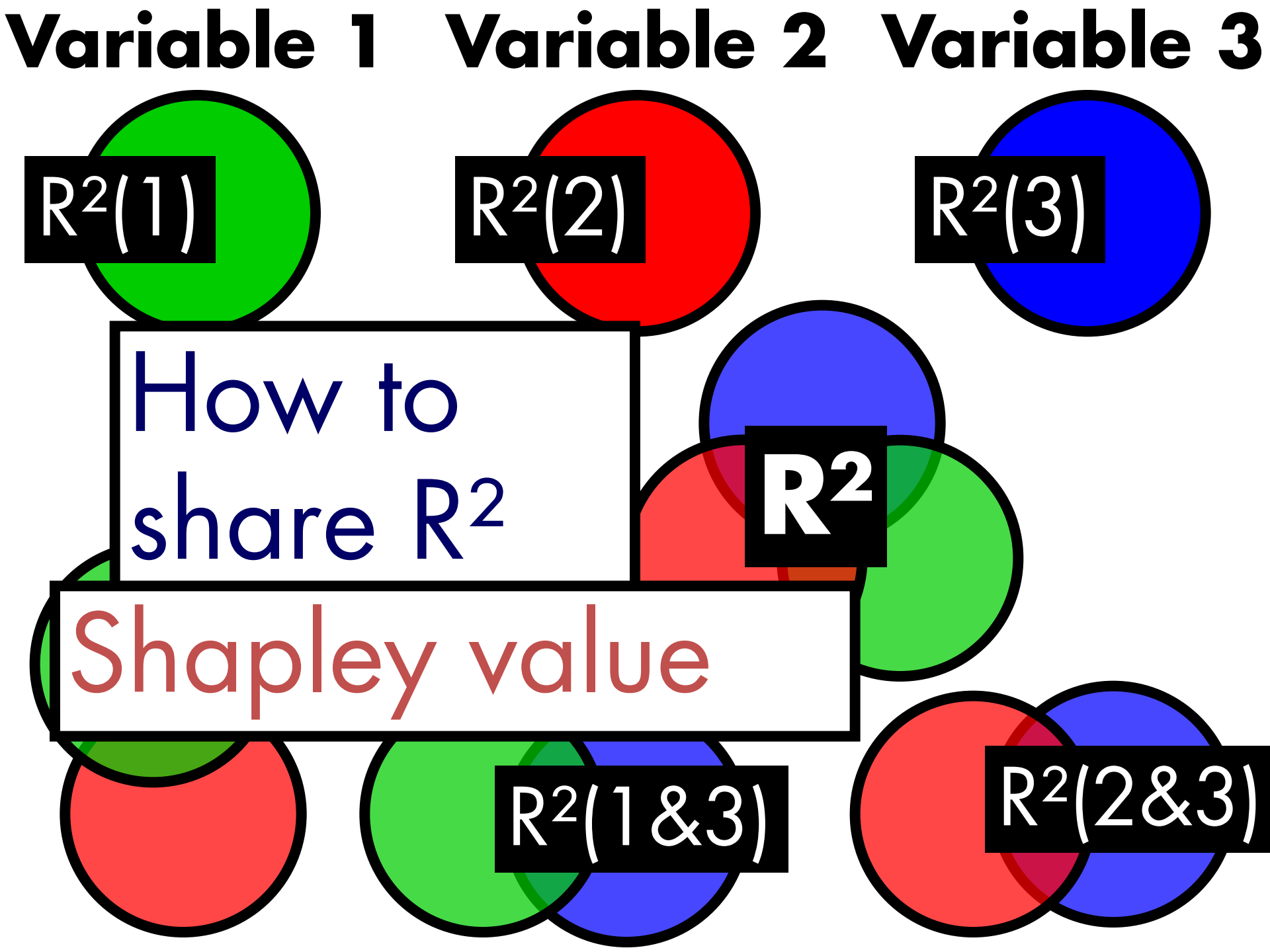

Variable 1
Variable 2
Variable 3
R²(1)
R²(2)
R²(3)
R²
R²(1&2)
R²(1&3)
R²(2&3)


Variable 1
Variable 2
Variable 3
R²(1)
R²(2)
R²(3)
How to share R²
R²
Shapley value
R²(1&3)
R²(2&3)

How to decompose R^2

- For STATA

`http://www.uni-leipzig.de/~rego`

standard

```
. reg mpg weight length headroom price
```

Source	SS	df	MS
Model	1629.24686	4	407.311716
Residual	814.212596	69	11.8001826
Total	2443.45946	73	33.4720474

Number of obs	74
F(4, 69)	34.52
Prob > F	0.0000
R-squared	0.6668
Adj R-squared	0.6475
Root MSE	3.4351

mpg	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
weight	-.0030149	.0017844	-1.69	0.096	-.0065746	.0005448
length	-.0942241	.0588767	-1.60	0.114	-.2116799	.0232316
headroom	-.1387698	.5611849	-0.25	0.805	-1.258303	.9807635
price	-.0001795	.0001706	-1.05	0.296	-.00052	.0001609
_cons	49.6308	6.375722	7.78	0.000	36.91158	62.35002

standard

```
. reg mpg weight length headroom price
```

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mpg	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
weight	-.0030149	.0017844	-1.69	0.096	-.0065746	.0005448
length	-.0942241	.0588767	-1.60	0.114	-.2116799	.0232316
headroom	-.1387698	.5611849	-0.25	0.805	-1.258303	.9807635
price	-.0001795	.0001706	-1.05	0.296	-.00052	.0001609
_cons	49.6308	6.375722	7.78	0.000	36.91158	62.35002

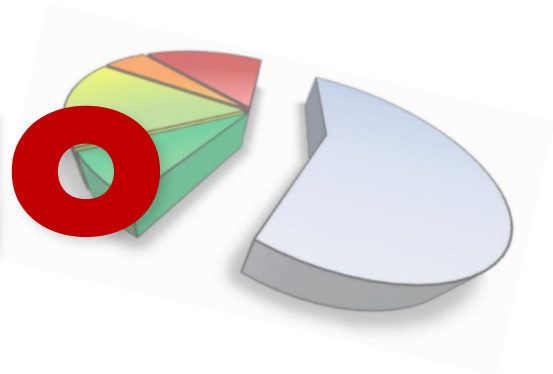
rego

```
. rego mpg weight length headroom price
```

Regressor	Coef.		Std.Err.	P>|t|	Std.Coef.	Shapley %R2
weight	-.0030149	*	.0017844	0.096	-0.4050	41.0462
length	-.0942241		.0588767	0.114	-0.3626	39.5705
headroom	-.1387698		.5611849	0.805	-0.0203	8.1060
price	-.0001795		.0001706	0.296	-0.0915	11.2773
Intercept	49.6308		6.375722	0.000		
Observations	74					
Overall R2	0.66678					
Root MSE	3.435139					
F-stat. Model	34.51741	***		0.000		
Log Likelihood	-193.7332					

Decomposing R^2 with the Owen Value

rego



Cooperative Game Theory

Player's joint Possibilities

Assume: they work together

Formula

Shapley value

Individual payoffs

Cooperative Game Theory

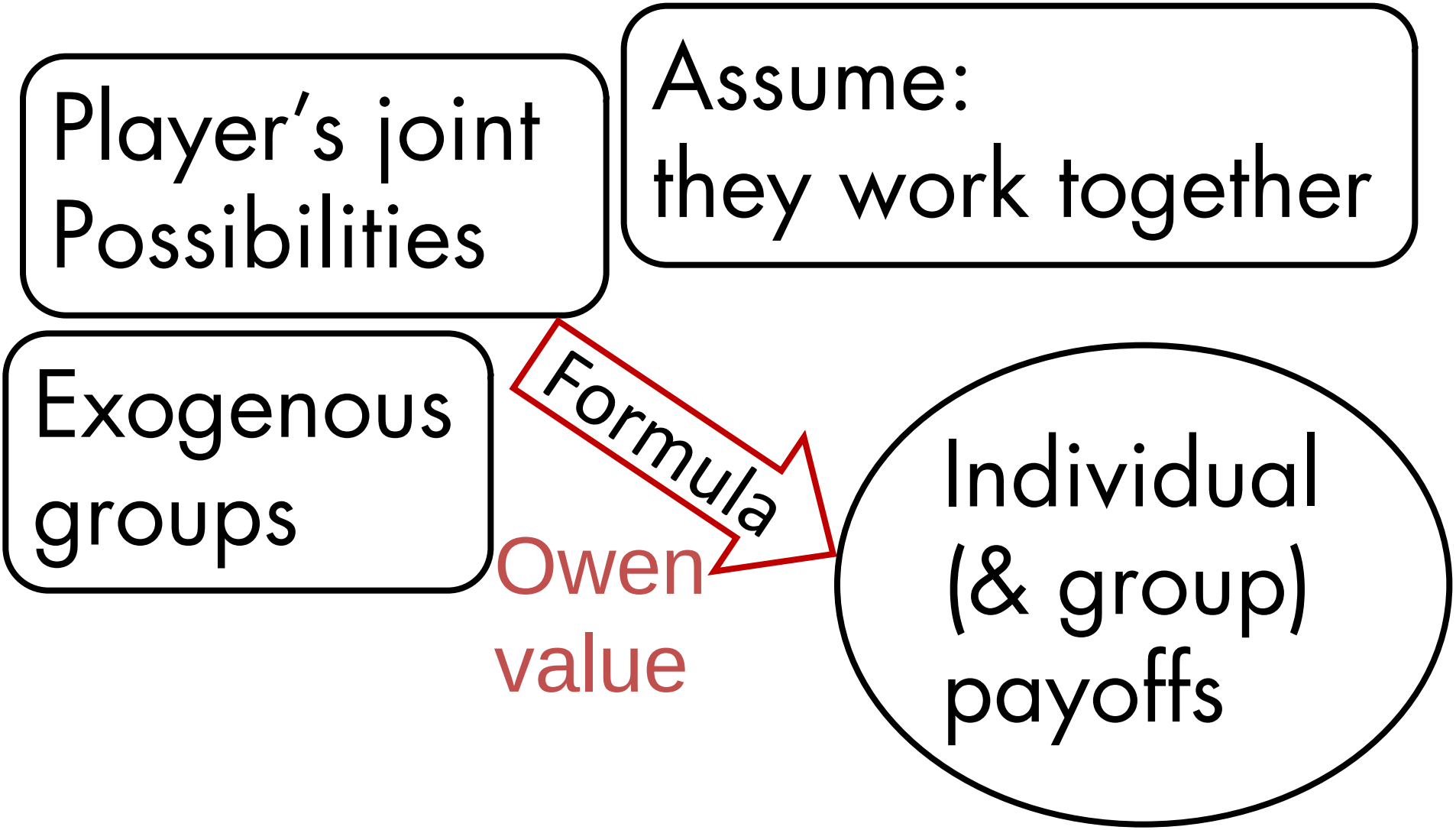
Player's joint Possibilities

Assume: they work together

Exogenous groups

Individual (& group) payoffs

Cooperative Game Theory



The Zoo

- Ape Bat Chameleon

- Groups: Mamals Reptiles

Shapley & Owen value

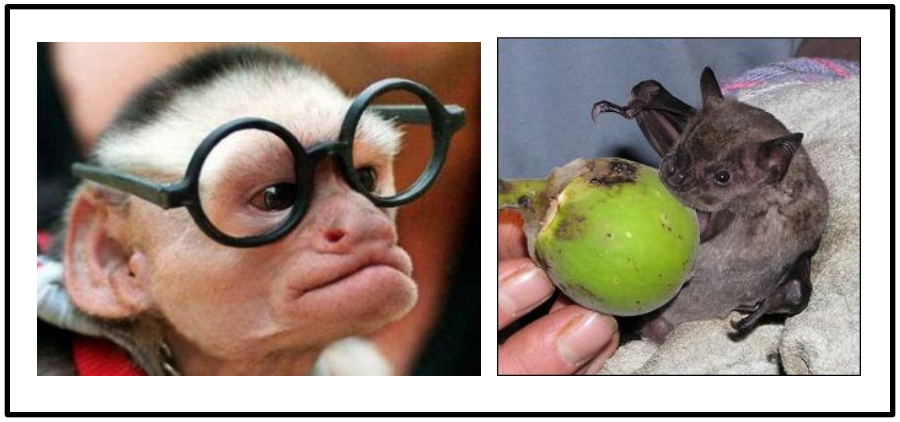
$$Sh_A = \frac{1}{\#\boldsymbol{R}} \cdot \sum_{\mathbf{r} \in \boldsymbol{R}} MC_A(\mathbf{r})$$

$$Ow_A = \frac{1}{\#\boldsymbol{R}_P} \cdot \sum_{\mathbf{r} \in \boldsymbol{R}_P} MC_A(\mathbf{r})$$

$\boldsymbol{R}_P$ contains only rank orderings respecting the partition

1. List all permutations of players

A, B, C
~~A, C, B~~
B, A, C
~~B, C, A~~
C, A, B
C, B, A



24

12

18

$Ow_A = 34.5$ $Ow_B = 16.5$ $Ow_C = 21$

$Sh_A = 34$ $Sh_B = 16$ $Sh_C = 22$

54

48

30

Why Owen value?

THEOREM (Khm & Yan 2007):

Only one solution satisfies

Monotonicity

+ treats equal players equally

+ treats equal groups equally.

It is the Owen value.

Why Owen value?

THEOREM (Khm & Yan 2007):

Only one solution satisfies

Monotonicity

+ treats equal players equally (°)

+ treats equal groups equally.

It is the Owen value.

(°) players from different groups are now unequal

Exogenous groups for OLS

- Polynomials (exp, exp^2)
- Dummy variables
- Conceptually related variables
  - By theory (Solow model)
  - By notion (socio-economic status; savings and investments)

Income =

			R^2 decomposition (%)	
Group	Regressor	Coef.	Owen	Group
1	SCT	0.789 *	3.0	33.2
	SCT × EDUC	-0.048 *	8.3	
	EDUC	0.103 ***	21.9	
2	EXPER	0.025 ***	7.0	11.0
	$(\text{EXPER})^2/100$	-0.041 ***	4.0	
3	TENURE	0.017 ***	9.3	14.3
	$(\text{TENURE})^2/100$	-0.029 **	5.0	
4	MARRIED	0.084 ***	5.0	5.0
5	Firm size	(3 dummies) ***		14.7
6	Industry	(6 dummies) ***		5.5
7	Region	(14 dummies) ***		16.2

Frank Huettner and Marco Sunder:

Axiomatic arguments for decompo-sing goodness of fit according to Shapley and Owen values

in:
Electronic Journal of Statistics, forthcoming.

Income =

			R^2 decomposition (%)	
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6	Industry	(6 dummies) ***		5.5
7	Region	(14 dummies) ***		16.2

THANK
YOU

Nice to have

$y = \beta_0$ → $R^2(\varnothing) = 0$